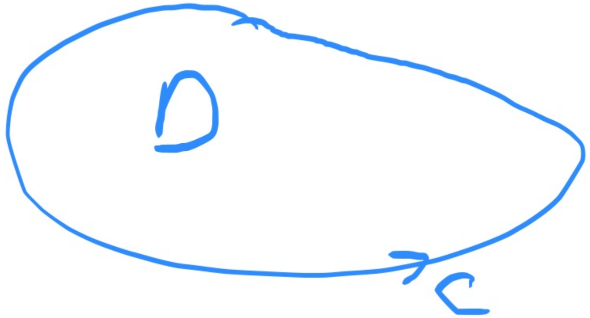
Green's Theorem



$F: \mathbb{R}^2 \to \mathbb{R}^2$

vector field

$F(x,y) = (P(x,y), Q(x,y))$

$$\Rightarrow \int_C F \cdot ds = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dx\, dy$$

relates a line integral of F

to a double integral "fancy derivative" of F

Stokes' Theorem: generalization of Green's Theorem

for surfaces embedded in $\mathbb{R}^3$

need: analog of "fancy derivative" for vector fields in $\mathbb{R}^3$

Recall: The del operator ∇ is the formal vector $(\partial/\partial x, \partial/\partial y, \partial/\partial z)$

Def let $F: \mathbb{R}^3 \to \mathbb{R}^3$ a vector field

Then curl F is the vector field defined by

$$\boxed{\operatorname{curl} F = \nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}}$$

where $F = (F_1, F_2, F_3)$

Example: Let $F(x,y,z) = (xy, \cos z, 1)$

Calculate cure $F = \nabla \times F$

Answer:

$$\text{cure } F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & \cos z & 1 \end{vmatrix} =$$

$$= \vec{i}\left(\partial/\partial y (1) - \partial/\partial z (\cos z) \right) - \vec{j} \left| \partial/\partial x (1) - \partial/\partial z (xy) \right|$$

$$+ \vec{k} \left(\partial/\partial x (\cos z) - \partial/\partial y (xy) \right))$$

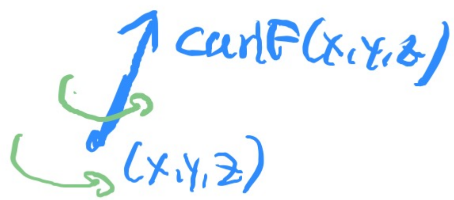
$$= \vec{i} \sin z - \vec{j}(0-0) + \vec{k}(0-x)$$

$$= \boxed{\vec{i} \sin z - \vec{k} x} \quad \text{or} \quad \boxed{\text{curl } F(x,y,z) = (\sin z, 0, -x)}$$

Physical interpretation of curl F

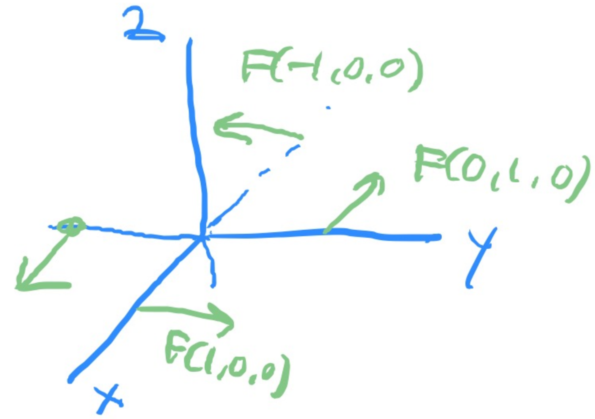
Assume F describes motion of a liquid

$\Rightarrow$ curl $F(x,y,z)$ describes local rotation of liquid.



direction of curl $F(x,y,z)$ = rotation axis
length of " = angular velocity.

simple example: $F(x,y,z) = (-y, x, 0)$



Calculate

$$\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial x & \partial/\partial z \\ -y & x & 0 \end{vmatrix}$$

$= (0, 0, 2)$ $\leftarrow$ rotation axis = z-axis

Special case:

$$F(x,y,z) = (P(x,y), Q(x,y), 0)$$

check: curl $F = \cdots\cdots$

$$= \left(0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

Def. For a 2-dim vector field $F(x,y) = (P(x,y), Q(x,y))$ the scalar curl of F is given by $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

Section 7.6 Surface integrals of vector fields

$S \subset \mathbb{R}^3$ a surface with parametrization $\underline{\Phi}: D \to \mathbb{R}^3$

$F: \mathbb{R}^3 \to \mathbb{R}^3$ vector field

Def $\iint_{\underline{\Phi}} F \cdot dS = \iint_D F(\underline{\Phi}(u,v)) \cdot (T_u \times T_v)\, du\, dv$

Theorem
- The surface integral is independent of the choice of parametrization $\underline{\Phi}$ up to a sign
- The sign depends on to which side of S the normal vector $T_u \times T_v$ points

Example Let S be the upper half sphere of radius 2 and let $F(x,y,z) = (0,0,z)$

Calculate $\iint_S F \cdot dS$ for a parametrization for which the normal vector points out of the sphere (Remark: This is the specified positive side).

Solution: use spherical coordinates:

$$\Phi(\phi,\theta) = (2\sin\phi\cos\theta,\ 2\sin\phi\sin\theta,\ 2\cos\phi) = (x,y,z)$$

$$T_\phi = (2\cos\phi\cos\theta,\ 2\cos\phi\sin\theta,\ -2\sin\phi)$$

$$T_\theta = (-2\sin\phi\sin\theta,\ 2\sin\phi\cos\theta,\ 0)$$

result $T_\phi \times T_\theta = \dots = 4\sin\phi(\sin\phi\cos\theta,\ \sin\phi\sin\theta,\ \cos\phi)$

$$= 2\sin\phi\,(x,y,z)$$

Useful general result:

If we parametrize (part of) a sphere of radius a via spherical coordinates $\boxed{\Phi(\phi,\theta) = (a\sin\phi\cos\theta, \ldots)}$

$$\Rightarrow \boxed{T_\phi \times T_\theta = a\sin\phi\,\Phi(\phi,\theta) = a\sin\phi\,(x,y,z)}$$

$T_\phi \times T_\theta = a^2\sin\phi\,(x,y,z)$

$(x,y,z) = \Phi(\phi,\theta)$

result: $\boxed{T_\phi \times T_\theta \text{ points out of sphere.}}$

$$\Rightarrow \iint_S F\cdot dS = \int_0^{2\pi}\int_0^{\pi/2} \underbrace{F(x,y,z)}_{\Phi(\phi,\theta)}\cdot(T_\phi\times T_\theta)\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} (0,0,z)\cdot 2\sin\phi\,(x,y,z)\,d\phi\,d\theta =$$

$$= \int_0^{2\pi} \int_0^{\pi/2} 2 \sin\phi \; z^2 \, d\phi \, d\theta \qquad z = 2\cos\phi$$

$$= \int_0^{2\pi} \int_0^{\pi/2} 2 \sin\phi \; 4\cos^2\phi \, d\phi \, d\theta =$$

$$= \int_0^{2\pi} -\frac{8}{3} \cos^3\phi \Big|_0^{\pi/2} d\theta$$

$$= \int_0^{2\pi} -\frac{8}{3}(0-1)\, d\theta = \frac{8}{3} \cdot 2\pi = \boxed{\frac{16}{3}\pi}$$

homework: If $T(x,y,z)$ given (temperature)
temperature flux through S $= \iint_S \nabla T \cdot dS$